

# Relativistic Hydrodynamic Cosmological Perturbations

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Relativistic cosmological perturbation analyses can be made based on several different fundamental gauge conditions. In the pressureless limit the variables in certain gauge conditions show the correct Newtonian behaviors. Considering the general curvature ( $K$ ) and the cosmological constant ( $\Lambda$ ) in the background medium, the perturbed density in the comoving gauge, and the perturbed velocity and the perturbed potential in the zero-shear gauge show the same behavior as the Newtonian ones *in general scales*. In the first part, we elaborate these Newtonian correspondences. In the second part, using the identified gauge-invariant variables with correct Newtonian correspondences, we present the relativistic results with general pressures in the background and perturbation. We present the general *super-sound-horizon scale solutions* of the above mentioned variables valid for general  $K$ ,  $\Lambda$ , and generally evolving equation of state. We show that, for vanishing  $K$ , the super-sound-horizon scale evolution is characterised by a *conserved variable* which is the perturbed three-space curvature in the comoving gauge. We also present equations for the multi-component hydrodynamic situation and for the rotation and gravitational wave.

KEY WORDS: cosmology, gravitational instability, relativistic hydrodynamics

## I. INTRODUCTION

The analysis of gravitational instability in the expanding universe model was first presented by Lifshitz in 1946 in a general relativistic context [1]. Historically, the much simpler, and in hindsight, more intuitive Newtonian study followed later [2]. The pioneering study by Lifshitz is based on a gauge choice which is commonly called the synchronous gauge. As the later studies have shown, the synchronous gauge is only one way of fixing the gauge freedom out of several available fundamental gauge conditions [3–8]. As will be summarized in the following, out of the several gauge conditions only the synchronous gauge fails to fix the gauge mode completely and thus often needs more involved algebra. As long as one is careful of the algebra this gauge choice does not cause any kind of intrinsic problem; there exist, however, some persisting algebraic errors in the literature, see Sec. V. The gauge condition which turns out to be especially suitable for handling the perturbed density is the comoving gauge used first by Nariai in 1969 [4]. Since the comoving gauge condition completely fixes the gauge transformation property, the variables in this gauge can be equivalently considered as gauge-invariant ones. As mentioned, there exist several such fundamental gauge conditions each of which completely fixes the gauge transformation properties. One of such gauge conditions suitable for handling the gravitational potential and the velocity perturbations is the zero-shear gauge used first by Harrison in 1967 [3]. The variables in such gauge conditions are equivalently gauge-invariant. Using the gauge freedom as an advantage for handling the problem was emphasized by Bardeen in 1988 [9]. In order to use the gauge choice as an advantage a *gauge ready method* was proposed in [8] which will be adopted in the following, see Sec. II B.

The variables which characterize the self gravitating Newtonian fluid flow are the density, the velocity and the gravitational potential (the pressure is specified by an equation of state), whereas, the relativistic flow may be characterized by various components of the metric (and consequent curvatures) and the energy-momentum tensor. Since the relativistic gravity theory is a constrained system we have the freedom of imposing certain conditions on the metric or the energy-momentum tensor as coordinate conditions. In the perturbation analyses the freedom arises because we need to introduce a fictitious background system in order to describe the physical perturbed system. The correspondence of a given spacetime point between the perturbed spacetime and the fictitious background one could be made with certain degrees of freedom. This freedom can be fixed by suitable conditions (the gauge conditions) based on the spacetime coordinate transformation. Studies in the literature show that a certain variable in a certain gauge condition correctly reproduces the corresponding Newtonian behavior. Although the perturbed density in the comoving gauge shows the Newtonian behavior, the perturbed potential and the perturbed velocity in the same gauge do not behave like the Newtonian ones; for example, in the comoving gauge the perturbed velocity vanishes by the coordinate (gauge) condition. It is known that both the perturbed potential and the perturbed velocity in the zero-shear gauge correctly behave like the corresponding Newtonian variables [5].

In the first part (Sec. III) we will elaborate establishing such correspondences between the relativistic and Newtonian perturbed variables. Our previous work on this subject is presented in [10,11]; compared with [10] in this work we will explicitly compare the relativistic equations for the perturbed density, potential and velocity variables in several available gauge conditions with the ones in the Newtonian system. We will include both the background spatial curvature ( $K$ ) and the cosmological constant ( $\Lambda$ ). In the second part (Sec. IV), using the variables with correct Newtonian correspondences, we will extend our relativistic results to the situations with general pressures in the background and perturbations. We will present the relativistic equations satisfied by the gauge-invariant variables and will derive the *general solutions* valid in the super-sound-horizon scale (i.e., larger than Jeans scale) considering both  $K$  and  $\Lambda$ , and the generally evolving ideal fluid equation of state  $p = p(\mu)$ .

Section II is a review of our previous work displaying the basic equations in both Newtonian and relativistic contexts and summarizing our strategy for handling the equations. In Sec. III we consider a pressureless limit of the relativistic equations. We derive the equations for the perturbed density, the perturbed potential and the perturbed velocity in several different fundamental gauge conditions. By comparing these relativistic equations in several gauges with the Newtonian ones we *identify* the gauge conditions which reproduce the correct Newtonian behavior for certain variables in general scales. In Sec. IV we present the fully relativistic equations for the identified gauge-invariant variables with correct Newtonian limits, now, considering the general pressures in the background and perturbations. We derive the general large-scale solutions valid for general  $K$  and  $\Lambda$  in an ideal fluid medium, which are thus valid for general equation of state of the form  $p = p(\mu)$ , but with negligible entropic and anisotropic pressures. The solutions are valid in the super-sound-horizon scale, thus are valid effectively in all scales in the matter dominated era where the sound-horizon (Jeans scale) is negligible. We discuss several quantities which are conserved in the large-scale under the changing background equation of state. These are the perturbed three-space curvature in several gauge conditions, and in particular, we find the three-space curvature perturbation in the comoving gauge shows a distinguished behavior: for  $K = 0$  (but for general  $\Lambda$ ), it is conserved in a super-sound-horizon scale independently of the changing equation of state. For completeness we also summarize the case with multiple fluid components in Sec. IV C, and cases of the rotation and the gravitational wave in Sec. IV D. Sec. IV is the highlight of the present work. Sec. V is a discussion. We set  $c \equiv 1$ .

## II. BASIC EQUATIONS AND STRATEGY

### A. Newtonian Cosmological Perturbations

Since the Newtonian perturbation analysis in the expanding medium is well known we begin by summarizing the basic equations [10,12,13]. The background evolution is governed by

$$H^2 = \frac{8\pi G}{3}\varrho - \frac{K}{a^2} + \frac{\Lambda}{3}, \quad \varrho \propto a^{-3}, \quad (1)$$

where we allowed the general curvature (total energy) and the cosmological constant;  $a(t)$  is a cosmic scale factor,  $H(t) \equiv \dot{a}/a$ , and  $\varrho(t)$  is the mass density. In Newtonian theory the cosmological constant can be introduced in the Poisson equation *by hand* as  $\nabla^2\Phi = 4\pi G\varrho - \Lambda$ . Perturbed parts of the mass conservation, the momentum conservation and the Poisson's equations are [see Eqs. (43,46) in [10]]:

$$\delta\dot{\varrho} + 3H\delta\varrho = -\frac{k}{a}\varrho\delta v, \quad \delta\dot{v} + H\delta v = \frac{k}{a}\delta\Phi, \quad -\frac{k^2}{a^2}\delta\Phi = 4\pi G\delta\varrho, \quad (2)$$

where  $\delta\varrho(\mathbf{k}, t)$ ,  $\delta v(\mathbf{k}, t)$  and  $\delta\Phi(\mathbf{k}, t)$  are the Fourier modes of the perturbed mass density, velocity, and gravitational potential, respectively [see Eqs. (44,45) in [10]];  $k$  is a comoving wave number with  $\nabla^2 \rightarrow -k^2/a^2$ . [For linear perturbations the same forms of equations are valid in the configuration and the Fourier spaces. Thus, without causing any confusion, we often ignore distinguishing the Fourier space from the configuration space by an additional subindex.] Equation (2) can be arranged into the closed form equations for  $\delta$  ( $\equiv \delta\varrho/\varrho$ ),  $\delta v$  and  $\delta\Phi$  as:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\varrho\delta = \frac{1}{a^2H} \left[ a^2H^2 \left( \frac{\delta}{H} \right) \right]' = 0, \quad (3)$$

$$\delta\ddot{v} + 3H\delta\dot{v} + \left( \dot{H} + 2H^2 - 4\pi G\varrho \right) \delta v = 0, \quad (4)$$

$$\delta\ddot{\Phi} + 4H\delta\dot{\Phi} + \left( \dot{H} + 3H^2 - 4\pi G\varrho \right) \delta\Phi = \frac{1}{a^3H} \left[ a^2H^2 \left( \frac{a}{H}\delta\Phi \right) \right]' = 0. \quad (5)$$

We note that Eqs. (2-5) are valid for general  $K$  and  $\Lambda$ . Although redundant, we presented these equations for later comparison with the relativistic results. The general solutions for  $\delta$ ,  $\delta v$  and  $\delta\Phi$  immediately follow as (see also Table 1 in [10]):

$$\delta(\mathbf{k}, t) = k^2 \left[ HC(\mathbf{k}) \int_0^t \frac{dt}{\dot{a}^2} + \frac{H}{4\pi G \rho a^3} d(\mathbf{k}) \right], \quad (6)$$

$$\delta v(\mathbf{k}, t) = - \left[ \frac{k}{aH} C(\mathbf{k}) \left( 1 + a^2 H \dot{H} \int_0^t \frac{dt}{\dot{a}^2} \right) + \frac{k \dot{H}}{4\pi G \rho a^2} d(\mathbf{k}) \right], \quad (7)$$

$$\delta\Phi(\mathbf{k}, t) = - \left[ 4\pi G \rho a^2 HC(\mathbf{k}) \int_0^t \frac{dt}{\dot{a}^2} + \frac{H}{a} d(\mathbf{k}) \right]. \quad (8)$$

The coefficients  $C(\mathbf{k})$  and  $d(\mathbf{k})$  are two integration constants, and indicate the relatively growing and decaying solutions, respectively; the coefficients are matched in accordance with the solutions with a general pressure in Eqs. (47-49) using Eq. (39) (different dummy variables for  $t$  in the integrands are assumed). In the literature we can find various analytic form expressions of the above solutions in diverse situations, see [1,3,4,12-14].

## B. Relativistic Cosmological Perturbations

In the relativistic theory the fundamental variables are the metric and the energy-momentum tensor. As a way of summarizing our notation we present our convention of the metric and the energy-momentum tensor. As the metric we consider a spatially homogeneous and isotropic spacetime with the most general perturbation

$$ds^2 = -a^2 (1 + 2\alpha) d\eta^2 - a^2 (\beta_{,\alpha} + B_\alpha) d\eta dx^\alpha + a^2 \left[ g_{\alpha\beta}^{(3)} (1 + 2\varphi) + 2\gamma_{|\alpha\beta} + 2C_{(\alpha|\beta)} + 2C_{\alpha\beta} \right] dx^\alpha dx^\beta, \quad (9)$$

where  $0 = \eta$  with  $dt \equiv a d\eta$ , Indices  $\alpha, \beta \dots$  are spatial ones and  $g_{\alpha\beta}^{(3)}$  is the three-space metric of the homogeneous and isotropic space.  $\alpha(\mathbf{x}, t)$ ,  $\beta(\mathbf{x}, t)$ ,  $\gamma(\mathbf{x}, t)$ , and  $\varphi(\mathbf{x}, t)$  indicate the scalar-type structure with four degrees of freedom. The transverse  $B_\alpha(\mathbf{x}, t)$  and  $C_\alpha(\mathbf{x}, t)$  indicate the vector-type structure with four degrees of freedom. The transverse-tracefree  $C_{\alpha\beta}(\mathbf{x}, t)$  indicates the tensor-type structure with two degrees of freedom. The energy-momentum tensor is decomposed as:

$$\begin{aligned} T_0^0 &\equiv -\bar{\mu} - \delta\mu, & T_\alpha^0 &\equiv -\frac{1}{k} (\bar{\mu} + \bar{p}) v_{,\alpha} + \delta T^{(v)0}_\alpha, \\ T_\beta^\alpha &\equiv (\bar{p} + \delta p) \delta_\beta^\alpha + \frac{1}{a^2} \left( \nabla^\alpha \nabla_\beta - \frac{1}{3} \Delta \delta_\beta^\alpha \right) \sigma + \delta T^{(v)\alpha}_\beta + \delta T^{(t)\alpha}_\beta, \end{aligned} \quad (10)$$

where  $\mu(\mathbf{k}, t) \equiv \bar{\mu}(t) + \delta\mu(\mathbf{k}, t)$  and  $p \equiv \bar{p} + \delta p$  are the energy density and the pressure, respectively; an overbar indicates a background order quantity and will be ignored unless necessary.  $v$  is a frame-independent velocity related perturbed order variable and  $\sigma$  is an anisotropic pressure. Situation with multiple fluids will be considered in Sec. IV C.  $\delta T^{(v)0}_\alpha$ ,  $\delta T^{(v)\alpha}_\beta$ , and  $\delta T^{(t)\alpha}_\beta$  are vector and tensor-type perturbed order energy-momentum tensor. All the spatial tensor indices and the vertical bar are based on  $g_{\alpha\beta}^{(3)}$  as the metric. The three types of structures are related to the density condensation, the rotation, and the gravitational wave, respectively. Due to the high symmetry in the background the three types of perturbations decouple from each other to the linear order. Since these three types of structures evolve independently, we can handle them separately. Evolutions of the rotation and the gravitational wave will be discussed in Sec. IV D.

For the background equations we have [Eq. (6) in [15]]:

$$H^2 = \frac{8\pi G}{3} \mu - \frac{K}{a^2} + \frac{\Lambda}{3}, \quad \dot{H} = -4\pi G(\mu + p) + \frac{K}{a^2}, \quad \dot{\mu} = -3H(\mu + p). \quad (11)$$

These equations follow from  $G_0^0$  and  $G_\alpha^\alpha - 3G_0^0$  components of Einstein equations and  $T_{0;b}^b = 0$ , respectively; the third equation follows from the first two equations. For  $p = 0$  and replacing  $\mu$  with  $\rho$  Eq. (11) reduces to Eq. (1). The relativistic cosmological perturbation equations without fixing the temporal gauge condition, thus in a gauge ready form, were derived in Eq. (22-28) of [8] [see also Eqs. (8-14) in [15]]; we use  $\delta \equiv \delta\mu/\mu$  and  $v \equiv -(k/a)\Psi/(\mu + p)$ :

$$\kappa \equiv 3(-\dot{\varphi} + H\alpha) + \frac{k^2}{a^2}\chi, \quad (12)$$

$$-\frac{k^2 - 3K}{a^2}\varphi + H\kappa = -4\pi G\mu\delta, \quad (13)$$

$$\kappa - \frac{k^2 - 3K}{a^2}\chi = 12\pi G(\mu + p)\frac{a}{k}v, \quad (14)$$

$$\dot{\chi} + H\chi - \alpha - \varphi = 8\pi G\sigma, \quad (15)$$

$$\dot{\kappa} + 2H\kappa = \left(\frac{k^2}{a^2} - 3\dot{H}\right)\alpha + 4\pi G(1 + 3c_s^2)\mu\delta + 12\pi Ge, \quad (16)$$

$$\dot{\delta} + 3H(c_s^2 - w)\delta + 3H\frac{e}{\mu} = (1 + w)\left(\kappa - 3H\alpha - \frac{k}{a}v\right), \quad (17)$$

$$\dot{v} + (1 - 3c_s^2)Hv = \frac{k}{a}\alpha + \frac{k}{a(1 + w)}\left(c_s^2\delta + \frac{e}{\mu} - \frac{2}{3}\frac{k^2 - 3K}{a^2}\frac{\sigma}{\mu}\right), \quad (18)$$

where we introduced a spatially gauge-invariant combination  $\chi \equiv a(\beta + a\dot{\gamma})$ . According to their origins, Eqs. (12-18) can be called: the definition of  $\kappa$ , ADM energy constraint ( $G_0^0$  component of Einstein equations), momentum constraint ( $G_\alpha^0$  component), ADM propagation ( $G_\beta^\alpha - \frac{1}{3}\delta_\beta^\alpha G_\gamma^\gamma$  component), Raychaudhuri ( $G_\alpha^\alpha - G_0^0$  component), energy conservation ( $T_{0;b}^b = 0$ ), and momentum conservation ( $T_{\alpha;b}^b = 0$ ) equations, respectively. The perturbed metric variables  $\varphi(\mathbf{k}, t)$ ,  $\kappa(\mathbf{k}, t)$ ,  $\chi(\mathbf{k}, t)$  and  $\alpha(\mathbf{k}, t)$  are the perturbed part of the three space curvature, expansion, shear, and the lapse function, respectively; meanings for  $\varphi$ ,  $\kappa$ , and  $\chi$  are based on the normal frame vector field. The isotropic pressure is decomposed as

$$\delta p(\mathbf{k}, t) \equiv c_s^2(t)\delta\mu(\mathbf{k}, t) + e(\mathbf{k}, t), \quad c_s^2 \equiv \frac{\dot{p}}{\dot{\mu}}, \quad w(t) \equiv \frac{p}{\mu}. \quad (19)$$

Under the gauge transformation  $\tilde{x}^a = x^a + \xi^a$ , the variables transform as (see Sec. 2.2 in [8]):

$$\begin{aligned} \tilde{\alpha} &= \alpha - \dot{\xi}^t, & \tilde{\varphi} &= \varphi - H\xi^t, & \tilde{\chi} &= \chi - \xi^t, & \tilde{\kappa} &= \kappa + \left(3\dot{H} - \frac{k^2}{a^2}\right)\xi^t, \\ \tilde{v} &= v - \frac{k}{a}\xi^t, & \tilde{\delta} &= \delta + 3(1 + w)H\xi^t, \end{aligned} \quad (20)$$

where  $\xi^t \equiv e^s \xi^s$ , and  $e$  and  $\sigma$  are gauge-invariant. [Due to the spatial homogeneity of the background, the effect from the spatial gauge transformation has been trivially handled;  $\chi$  and  $v$  are the spatially gauge-invariant combinations of the variables, and the other metric and fluid variables are naturally spatially gauge-invariant [9].] Thus, the perturbed variables in Eqs. (12-18) are *designed* so that any one of the following conditions can be used to fix the freedom based on the temporal gauge transformation:  $\alpha \equiv 0$  (the synchronous gauge),  $\varphi \equiv 0$  (the uniform-curvature gauge),  $\chi \equiv 0$  (the zero-shear gauge),  $\kappa \equiv 0$  (the uniform-expansion gauge),  $v/k \equiv 0$  (the comoving gauge), and  $\delta \equiv 0$  (the uniform-density gauge). These gauge conditions contain most of the gauge conditions used in the literature concerning the cosmological hydrodynamic perturbation in a single component situation; for the multiple component case see Eq. (69). Of course, we can make an infinite number of different linear combinations of these gauge conditions each of which is also suitable for the temporal gauge fixing condition. Our reason for choosing these conditions as fundamental is partly based on conventional use, but apparently, as the names of the conditions indicate, the gauge conditions also have reasonable meanings based on the metric and the energy-momentum tensors.

Each one of these gauge conditions, except for the synchronous gauge, fixes the temporal gauge transformation property completely. Thus, each variable in these five gauge conditions is equivalent to a corresponding gauge-invariant combination (of the variable concerned and the variable used in the gauge condition). We proposed a convenient way of writing the gauge-invariant variables [8]. For example, we let

$$\delta_v \equiv \delta + 3(1 + w)\frac{aH}{k}v \equiv 3(1 + w)\frac{aH}{k}v_\delta, \quad \varphi_\chi \equiv \varphi - H\chi \equiv -H\chi_\varphi, \quad v_\chi \equiv v - \frac{k}{a}\chi \equiv -\frac{k}{a}\chi_v. \quad (21)$$

The variables  $\delta_v$ ,  $v_\chi$  and  $\varphi_\chi$  are gauge-invariant combinations;  $\delta_v$  becomes  $\delta$  in the comoving gauge which takes  $v/k = 0$  as the gauge condition, etc. In this manner we can systematically construct the corresponding gauge-invariant combination for any variable based on a gauge condition which fixes the temporal gauge transformation property completely. One variable evaluated in different gauges can be considered as different variables, and they show different behaviors in general. In this sense, except for the synchronous gauge, the variables in the rest of five

gauges can be considered as the gauge-invariant variables. [The variables with a subindex  $\alpha$  are not gauge-invariant, because those are equivalent to variables in the synchronous gauge.] Although  $\delta_v$  is a gauge-invariant variable which becomes  $\delta$  in the comoving gauge,  $\delta_v$  itself can be evaluated in any gauge with the same value.

Equations (12-18) are the basic set of gauge-ready form equations for hydrodynamic perturbations proposed in [8]: “The moral is that one should work in the gauge that is mathematically most convenient for the problem at hand”, [9]. Accordingly, in order to use the available gauge conditions as the advantage, Eqs. (12-18) are designed for easy implementation of the fundamental gauge conditions. Using this *gauge-ready formulation*, complete sets of solutions for the six different gauge conditions are presented in an ideal fluid case [15], and in a pressureless medium [10]. Equations (11-19) include general pressures which may account for the nonequilibrium or dissipative effects in hydrodynamic flows in the cosmological context with general  $K$  and  $\Lambda$ . The equations are expressed in general forms so that the fluid quantities can represent successfully the effects of the scalar field and classes of generalized gravity theories [8]. The applications of the gauge-ready formalism to the minimally coupled scalar field and to the generalized gravity theories have been made in [16].

### III. NEWTONIAN CORRESPONDENCE OF THE RELATIVISTIC ANALYSES

In [10,11] we developed arguments on the correspondences between the Newtonian and the relativistic analyses. In order to reinforce the Newtonian correspondences of certain (gauge-invariant) variables in certain gauges, in the following we will present the closed form differential equations for  $\delta$ ,  $v$  and  $\varphi$  in the six different gauge conditions and compare them with the Newtonian ones in Eqs. (3-5).

There are reasons why we should know about possible Newtonian behaviors of the relativistic variables. Variables in the relativistic gravity are only parameters appearing in the metric and energy-momentum tensor. For example, later we will find that only in the zero-shear gauge does the perturbed curvature variable behave as the perturbed Newtonian potential, which we have some experience of in Newtonian physics. The same variable evaluated in other gauge conditions is simply other variable and we cannot regard it as the perturbed potential. As we have introduced several different fundamental gauge conditions it became necessary to identify the correct gauge where a variable shows the corresponding Newtonian behavior. As will be shown in this section, we rarely find Newtonian correspondences. In fact, there exists an almost unique gauge condition for each relativistic perturbation variable which shows Newtonian behavior. The results will be summarized in Sec. III G.

In this section we consider a *pressureless medium*,  $p = 0$  (thus  $w = 0 = c_s^2$ ) and  $e = 0 = \sigma$ . However, we consider general  $K$  and  $\Lambda$ .

#### A. Comoving Gauge

As the gauge condition we set  $v/k \equiv 0$ . Equivalently, we can set  $v/k = 0$  and leave the other variables as the gauge-invariant combinations with subindices  $v/k$  or simply  $v$ . From Eq. (18) we have  $\alpha_v = 0$ . Thus the comoving gauge is a case of the synchronous gauge; this is true *only* in a pressureless situation. From Eqs. (12-18) we can derive:

$$\ddot{\delta}_v + 2H\dot{\delta}_v - 4\pi G\mu\delta_v = 0, \quad (22)$$

$$\ddot{\varphi}_v + 3H\dot{\varphi}_v - \frac{K}{a^2}\varphi_v = 0. \quad (23)$$

Thus, Eq. (22) has a form identical to Eq. (3). However, Eq. (23) differs from Eq. (5), and apparently we do not have an equation for  $v$ .

For  $K = 0$  Eq. (23) leads to two solutions which are  $\varphi_v \propto \text{constant}$  and  $\int_0^t a^{-3} dt$ . Since we have  $\alpha_v = 0$ , for  $K = 0$ , from Eqs. (12,14) we have  $\dot{\varphi}_v = 0$ . This implies that, for  $K = 0$ , the second solution,  $\varphi_v \propto \int_0^t a^{-3} dt$ , should have the vanishing coefficient, see Eq. (56). The solution of Eq. (23) for general  $K$  will be presented later in Eq. (53).

#### B. Synchronous Gauge

We let  $\alpha \equiv 0$ . This gauge condition does not fix the temporal gauge transformation property completely. Thus, although we can still indicate the variables in this gauge condition using subindices  $\alpha$  without ambiguity, these variables are not gauge-invariant; see the discussion after Eq. (52). Equation (18) leads to

$$v_\alpha = c_g \frac{k}{a}, \quad (24)$$

which is a pure gauge mode. Thus, fixing  $c_g \equiv 0$  exactly corresponds to the comoving gauge. We can show that the following two equations are not affected by the remaining gauge mode.

$$\ddot{\delta}_\alpha + 2H\dot{\delta}_\alpha - 4\pi G\mu\delta_\alpha = 0, \quad (25)$$

$$\ddot{\varphi}_\alpha + 3H\dot{\varphi}_\alpha - \frac{K}{a^2}\varphi_\alpha = 0. \quad (26)$$

Equation (25) is identical to Eq. (3). This is because in a pressureless medium the behavior of the gauge modes happen to coincide with the behavior of the decaying physical solutions for  $\delta_\alpha$  and  $\varphi_\alpha$ . However, for the variables  $\kappa_\alpha$  and  $\chi_\alpha$  the gauge mode contribution appears explicitly.

Thus, in a pressureless medium, variables in the synchronous gauge behave the same as the ones in the comoving gauge, except for the additional gauge modes which appear in the synchronous gauge for some variables. In a pressureless medium, we can simultaneously impose both the comoving gauge and the synchronous gauge conditions. However, this is possible only in a pressureless medium, see the Appendix. In Sec. V we indicate some common errors in the literature based on the synchronous gauge.

### C. Zero-shear Gauge

We let  $\chi \equiv 0$ , and substitute the other variables into the gauge-invariant combinations with subindices  $\chi$ . We can derive:

$$\begin{aligned} \ddot{\delta}_\chi + \frac{2k^2/a^2 - 36\pi G\mu \left[1 + (H^2 - 2\dot{H})a^2/(k^2 - 3K)\right]}{k^2/a^2 - 12\pi G\mu \left[1 + 3H^2a^2/(k^2 - 3K)\right]} H \left(\dot{\delta}_\chi - 12\pi G\mu \frac{a^2}{k^2 - 3K} H\delta_\chi\right) \\ - 4\pi G\mu \left[1 - 3(H^2 - 2\dot{H})\frac{a^2}{k^2 - 3K}\right] \delta_\chi = 0, \end{aligned} \quad (27)$$

$$\ddot{v}_\chi + 3H\dot{v}_\chi + \left(\dot{H} + 2H^2 - 4\pi G\mu\right)v_\chi = 0, \quad (28)$$

$$\ddot{\varphi}_\chi + 4H\dot{\varphi}_\chi + \left(\dot{H} + 3H^2 - 4\pi G\mu\right)\varphi_\chi = 0. \quad (29)$$

Thus, Eqs. (28,29) have forms identical to Eqs. (4,5), respectively. However, only in the small-scale limit is the behavior of  $\delta_\chi$  the same as the Newtonian one.

### D. Uniform-expansion Gauge

We let  $\kappa \equiv 0$ , and substitute the remaining variables into the gauge-invariant combinations with subindices  $\kappa$ . We can derive:

$$\ddot{\delta}_\kappa + \left(2H - \frac{12\pi G\mu H}{k^2/a^2 - 3\dot{H}}\right)\dot{\delta}_\kappa - \left[4\pi G\mu \frac{k^2/a^2 + 6H^2}{k^2/a^2 - 3\dot{H}} + \left(\frac{12\pi G\mu H}{k^2/a^2 - 3\dot{H}}\right)\right]\delta_\kappa = 0, \quad (30)$$

$$\begin{aligned} \ddot{v}_\kappa + \left[2H - \frac{12\pi G\mu H}{k^2/a^2 - 3\dot{H}} + \frac{4\pi G\mu}{k^2/a^2 - 3\dot{H}} \left(\frac{k^2/a^2 - 3\dot{H}}{4\pi G\mu}\right)\right]\dot{v}_\kappa \\ + \left[\dot{H} + H^2 - 4\pi G\mu \frac{k^2/a^2 + 3H^2}{k^2/a^2 - 3\dot{H}} + H \frac{4\pi G\mu}{k^2/a^2 - 3\dot{H}} \left(\frac{k^2/a^2 - 3\dot{H}}{4\pi G\mu}\right)\right]v_\kappa = 0, \end{aligned} \quad (31)$$

$$\ddot{\varphi}_\kappa + \left(4H - \frac{12\pi G\mu H}{k^2/a^2 - 3\dot{H}}\right)\dot{\varphi}_\kappa + \left[\dot{H} + 3H^2 - 4\pi G\mu \frac{k^2/a^2 + 9H^2}{k^2/a^2 - 3\dot{H}} - \left(\frac{12\pi G\mu H}{k^2/a^2 - 3\dot{H}}\right)\right]\varphi_\kappa = 0. \quad (32)$$

In the small-scale limit we can show that Eqs. (30-32) reduce to Eqs. (3-5), respectively. Thus, *in the small-scale limit*, all three variables  $\delta_\kappa$ ,  $v_\kappa$  and  $\varphi_\kappa$  correctly reproduce the Newtonian behavior. However, outside or near the (visual) horizon scale, the behaviors of all these variables *strongly deviate* from the Newtonian ones.

In Sec. 84 of [13] we find that in order to get the usual Newtonian equations a coordinate transformation is made so that we have  $\dot{h} \equiv 0$  in the new coordinate. We can show that  $\dot{h} = 2\kappa$  in our notation. Thus, the new coordinate used in [13] is the uniform-expansion gauge<sup>1</sup>.

### E. Uniform-curvature Gauge

We let  $\varphi \equiv 0$ , and substitute the other variables into the gauge-invariant combinations with subindices  $\varphi$ . We have

$$\ddot{\delta}_\varphi + 2H \frac{(k^2 - 3K)/a^2 + 18\pi G\mu}{(k^2 - 3K)/a^2 + 12\pi G\mu} \dot{\delta}_\varphi - \frac{4\pi G\mu k^2/a^2}{(k^2 - 3K)/a^2 + 12\pi G\mu} \delta_\varphi = 0, \quad (33)$$

$$\ddot{v}_\varphi + \left(5H + 2\frac{\dot{H}}{H}\right) \dot{v}_\varphi + \left(3\dot{H} + 4H^2\right) v_\varphi = 0. \quad (34)$$

In the small-scale limit Eq. (33) reduces to Eq. (3). In the uniform-curvature gauge, the perturbed potential is set to be equal to zero by the gauge condition. The uniform-curvature gauge condition is known to have distinct properties in handling the scalar field or the dilaton field which appears in a broad class of the generalized gravity theories [16,21].

### F. Uniform-density Gauge

We let  $\delta \equiv 0$ , and substitute the other variables into the gauge-invariant combinations with subindices  $\delta$ . We have

$$\ddot{v}_\delta + 2\left(2H + \frac{\dot{H}}{H}\right) \dot{v}_\delta + 3\left(H^2 + \dot{H}\right) v_\delta = 0, \quad (35)$$

$$\ddot{\varphi}_\delta + 2H \frac{(k^2 - 3K)/a^2 + 18\pi G\mu}{(k^2 - 3K)/a^2 + 12\pi G\mu} \dot{\varphi}_\delta - \frac{4\pi G\mu k^2/a^2}{(k^2 - 3K)/a^2 + 12\pi G\mu} \varphi_\delta = 0. \quad (36)$$

These equations differ from Eqs. (4,5). Of course, we have no equation for  $\delta$  which is set to be equal to zero by our choice of the gauge condition.

### G. Newtonian Correspondences

After a thorough comparison made in this section we found that equations for  $\delta$  in the comoving gauge ( $\delta_v$ ), and for  $v$  and  $\varphi$  in the zero-shear gauge ( $v_\chi$  and  $\varphi_\chi$ ) show the same forms as the corresponding Newtonian equations *in general scales* and considering general  $K$  and  $\Lambda$ . Using these gauge-invariant combinations in Eq. (21), Eqs. (12-18) can be combined to give:

$$\dot{\delta}_v = -\frac{k^2 - 3K}{ak} v_\chi, \quad \dot{v}_\chi + H v_\chi = -\frac{k}{a} \varphi_\chi, \quad \frac{k^2 - 3K}{a^2} \varphi_\chi = 4\pi G\mu \delta_v. \quad (37)$$

Comparing Eq. (37) with Eq. (2) we can identify either one of the following correspondences:

$$\delta_v \leftrightarrow \delta|_{\text{Newtonian}}, \quad \frac{k^2 - 3K}{k^2} v_\chi \leftrightarrow \delta v|_{\text{Newtonian}}, \quad -\frac{k^2 - 3K}{k^2} \varphi_\chi \leftrightarrow \delta\Phi|_{\text{Newtonian}}, \quad (38)$$

$$\frac{k^2}{k^2 - 3K} \delta_v \leftrightarrow \delta|_{\text{Newtonian}}, \quad v_\chi \leftrightarrow \delta v|_{\text{Newtonian}}, \quad -\varphi_\chi \leftrightarrow \delta\Phi|_{\text{Newtonian}}. \quad (39)$$

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<sup>1</sup>This was incorrectly pointed out after Eq. (49) in [10]: In [10] it was wishfully mentioned that in Sec. 84 of [13] the gauge transformations were made into the comoving gauge for  $\delta$  and into the zero-shear gauge for  $v$  and  $\varphi$ , respectively.

In [10] we proposed the correspondences in Eq. (38), but the ones in Eq. (39) also work well. In fact, the gravitational potential identified in Eq. (39) is the one often found in the Newtonian limit of the general relativity, e.g. in [12]. Using Eqs. (21,14,15,37) we can also identify the following relations:

$$\delta_v = 3H \frac{a}{k} v_\delta, \quad v_\chi = -\frac{k}{a} \chi_v = -\frac{ak}{k^2 - 3K} \kappa_v, \quad \varphi_\chi = -\alpha_\chi = -H \chi_\varphi. \quad (40)$$

From a given solution known in one gauge we can derive all the rest of the solutions even in other gauge conditions. This can be done either by using the gauge-invariant combination of variables or directly through gauge transformations. The complete set of exact solutions in a pressureless medium is presented in [10]. From our study in this section and using the complete solutions presented in Tables 1 and 2 of [10] we can identify variables in certain gauges which correspond to the Newtonian ones. These are summarized in Table 1.

[[TABLE 1.]]

As mentioned earlier, all three variables in the uniform-expansion gauge show Newtonian correspondence in the small-scale; however all of them change the behaviors near and above the horizon scale. In the small-scale limit, except for the uniform-density gauge where  $\delta \equiv 0$ ,  $\delta$  in all gauge conditions behaves in the same way [5]. Meanwhile, since only  $\delta_v$  ( $\delta$  in the comoving gauge) shows the Newtonian behavior in a general scale, it may be natural to identify  $\delta_v$  as the one most closely resembling the Newtonian one.

Notice that, although we have horizons in the relativistic analysis the equations for  $\delta_v$ ,  $v_\chi$  and  $\varphi_\chi$  keep the same form as the corresponding Newtonian equations *in the general scale*. Considering this as an additional point we regard  $\delta_v$ ,  $v_\chi$  and  $\varphi_\chi$  as the one most closely corresponding to the Newtonian variables. This argument will become stronger as we consider the case with a general pressure in the next section.

## IV. RELATIVISTIC COSMOLOGICAL HYDRODYNAMICS

### A. General Equations and Large-scale Solutions

In the previous sections we have shown that the gauge-invariant combinations  $\delta_v$ ,  $v_\chi$  and  $\varphi_\chi$  behave most similarly to the Newtonian  $\delta \equiv \delta \varrho / \varrho$ ,  $\delta v$  and  $\delta \Phi$ , respectively. The equations remain the same in a general scale which includes the superhorizon scales in the relativistic situation considering general  $K$  and  $\Lambda$ . In this section, we will present the fully general relativistic equations for  $\delta_v$ ,  $v_\chi$  and  $\varphi_\chi$  including the effects of the general pressure terms. Equations (12-18) are the basic set of perturbation equations in a gauge-ready form.

From Eqs. (14,17,18), Eqs. (15,18), Eqs. (13,14), and Eqs. (12,14,15), we have, respectively:

$$\dot{\delta}_v - 3Hw\delta_v = -(1+w) \frac{k}{a} \frac{k^2 - 3K}{k^2} v_\chi - 2H \frac{k^2 - 3K}{a^2} \frac{\sigma}{\mu}, \quad (41)$$

$$\dot{v}_\chi + H v_\chi = -\frac{k}{a} \varphi_\chi + \frac{k}{a(1+w)} \left[ c_s^2 \delta_v + \frac{e}{\mu} - 8\pi G(1+w) \sigma - \frac{2}{3} \frac{k^2 - 3K}{a^2} \frac{\sigma}{\mu} \right], \quad (42)$$

$$\frac{k^2 - 3K}{a^2} \varphi_\chi = 4\pi G \mu \delta_v, \quad (43)$$

$$\dot{\varphi}_\chi + H \varphi_\chi = -4\pi G(\mu + p) \frac{a}{k} v_\chi - 8\pi G H \sigma. \quad (44)$$

Considering either one of the correspondences in Eqs. (38,39) we immediately see that Eqs. (41-44) have the correct Newtonian limit expressed in Eqs. (2,37). Equations (41-44) were presented in [5].

Combining Eqs. (41-44) we can derive closed form expressions for the  $\delta_v$  and  $\varphi_\chi$  which are the relativistic counterpart of Eqs. (3,5). We have:

$$\begin{aligned} \ddot{\delta}_v + (2 + 3c_s^2 - 6w) H \dot{\delta}_v + & \left[ c_s^2 \frac{k^2}{a^2} - 4\pi G \mu (1 - 6c_s^2 + 8w - 3w^2) + 12(w - c_s^2) \frac{K}{a^2} + (3c_s^2 - 5w) \Lambda \right] \delta_v \\ & = \frac{1+w}{a^2 H} \left[ \frac{H^2}{a(\mu + p)} \left( \frac{a^3 \mu}{H} \delta_v \right) \right] + c_s^2 \frac{k^2}{a^2} \delta_v \\ & = -\frac{k^2 - 3K}{a^2} \frac{1}{\mu} \left\{ e + 2H\dot{\sigma} + 2 \left[ -\frac{1}{3} \frac{k^2}{a^2} + 2\dot{H} + 3(1 + c_s^2) H^2 \right] \sigma \right\}, \end{aligned} \quad (45)$$



$$\begin{aligned} \ddot{\varphi}_\chi + (4 + 3c_s^2) H \dot{\varphi}_\chi + \left[ c_s^2 \frac{k^2}{a^2} + 8\pi G\mu (c_s^2 - w) - 2(1 + 3c_s^2) \frac{K}{a^2} + (1 + c_s^2) \Lambda \right] \varphi_\chi \\ = \frac{\mu + p}{H} \left[ \frac{H^2}{a(\mu + p)} \left( \frac{a}{H} \varphi_\chi \right) \right] + c_s^2 \frac{k^2}{a^2} \varphi_\chi = \text{stresses}. \end{aligned} \quad (46)$$

These two equations are related by Eq. (43) which resembles the Poisson equation. Notice the remarkably compact forms presented in the second steps of Eqs. (45,46). It may be an interesting exercise to show that the above equations are indeed valid for general  $K$  and  $\Lambda$ , and for the general equation of state  $p = p(\mu)$ ; use  $\dot{w} = -3H(1+w)(c_s^2 - w)$ . Equation (45) became widely known through Bardeen's seminal paper in [5]. In a less general context but originally Eqs. (45,46) were derived by Nariai [4] and Harrison [3], respectively; however, the compact expressions are new results in this paper.

If we ignore the entropic and anisotropic pressures ( $e = 0 = \sigma$ ) on scales larger than Jeans scale Eq. (46) immediately leads to a general integral form solution as

$$\begin{aligned} \varphi_\chi(\mathbf{k}, t) &= 4\pi G C(\mathbf{k}) \frac{H}{a} \int_0^t \frac{a(\mu + p)}{H^2} dt + \frac{H}{a} d(\mathbf{k}) \\ &= C(\mathbf{k}) \left[ 1 - \frac{H}{a} \int_0^t a \left( 1 - \frac{K}{\dot{a}^2} \right) dt \right] + \frac{H}{a} d(\mathbf{k}). \end{aligned} \quad (47)$$

This solution was first derived in Eq. (108) of [17]<sup>2</sup>, see also Eq. (55) in [8]. Solutions for  $\delta_v$  and  $v_\chi$  follow from Eqs. (43,44), respectively, as

$$\delta_v(\mathbf{k}, t) = \frac{k^2 - 3K}{4\pi G\mu a^2} \varphi_\chi(\mathbf{k}, t), \quad (48)$$

$$v_\chi(\mathbf{k}, t) = -\frac{k}{4\pi G(\mu + p)a^2} \left\{ C(\mathbf{k}) \left[ \frac{K}{\dot{a}} - \dot{H} \int_0^t a \left( 1 - \frac{K}{\dot{a}^2} \right) dt \right] + \dot{H} d(\mathbf{k}) \right\}. \quad (49)$$

We stress that these large-scale asymptotic solutions are *valid for the general  $K$ ,  $\Lambda$ , and  $p = p(\mu)$* ;  $C(\mathbf{k})$  and  $d(\mathbf{k})$  are integration constants considering the general evolution of  $p = p(\mu)$ . We also emphasize that *the large-scale criterion is the sound-horizon*, and thus in the matter dominated era the solutions are valid even far inside the (visual) horizon as long as the scales are larger than Jeans scale. In the pressureless limit, considering Eq. (39), Eqs. (47-49) correctly reproduce Eqs. (6-8).

Using the set of gauge-ready form equations in Eqs. (12-18) the complete set of corresponding general solutions for all variables in all different gauges can be easily derived; for such sets of solutions with less general assumptions see [10,15]. For  $K = 0 = \Lambda$  and  $w = \text{constant}$  we have  $a \propto t^{2/[3(1+w)]}$  and Eqs. (47-49) become:

$$\begin{aligned} \varphi_\chi(\mathbf{k}, t) &= \frac{3(1+w)}{5+3w} C(\mathbf{k}) + \frac{2}{3(1+w)} \frac{1}{at} d(\mathbf{k}) \\ &\propto \text{constant}, \quad t^{-\frac{5+3w}{3(1+w)}} \propto \text{constant}, \quad a^{-\frac{5+3w}{2}}, \\ \delta_v(\mathbf{k}, t) &\propto t^{\frac{2(1+3w)}{3(1+w)}}, \quad t^{-\frac{1-w}{1+w}} \propto a^{1+3w}, \quad a^{-\frac{3}{2}(1-w)}, \\ v_\chi(\mathbf{k}, t) &\propto t^{\frac{1+3w}{3(1+w)}}, \quad t^{-\frac{4}{3(1+w)}} \propto a^{\frac{1+3w}{2}}, \quad a^{-2}. \end{aligned} \quad (50)$$

These solutions were presented in [5] and in less general contexts but originally in [3,4]. For  $w = 0$  we recover the well known Newtonian behaviors.

## B. Conserved Quantities

In an ideal fluid, the curvature fluctuations in several gauge conditions are known to be conserved in the *superhorizon scale* independently of the changes in the background equation of state. From Eqs. (41,73) in [15] we find [for  $K = 0 = \Lambda$ , but for general  $p(\mu)$ ]:

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<sup>2</sup>We correct a typographical error in Eq. (129) of [17]:  $c_0/a$  should be replaced by  $c_0$ .

$$\varphi_v = \varphi_\delta = \varphi_\kappa = \varphi_\alpha = C(\mathbf{k}), \quad (51)$$

and the dominating decaying solutions vanish (or, are cancelled). The combinations follow from Eq. (20) as:

$$\varphi_v = \varphi - \frac{aH}{k}v, \quad \varphi_\delta \equiv \varphi + \frac{\delta}{3(1+w)} \equiv \zeta, \quad \varphi_\kappa \equiv \varphi + \frac{H}{3\dot{H} - k^2/a^2}\kappa, \quad \varphi_\alpha \equiv \varphi - H \int^t \alpha dt. \quad (52)$$

The  $\varphi_\alpha$  combination which is  $\varphi$  in the synchronous gauge is not gauge-invariant; the lower bound of integration gives the behavior of the gauge mode; in Eq. (51) we ignored the gauge mode. The large-scale conserved quantity,  $\zeta$ , proposed in [9,18] is  $\varphi_\delta$ . In [15] the above conservation properties are shown assuming  $K = 0 = \Lambda$ ; in such a case the integral form solutions for a complete set of variables are presented in Table 8 in [15]. In Eqs. (47-49) we have the large-scale integral form solutions in the case of general  $K$  and  $\Lambda$ . Thus, now we can see the behavior of these variables considering the general  $K$  and  $\Lambda$ .

Evaluating  $\varphi_v$  in Eq. (52) in the zero-shear gauge, thus  $\varphi_v = \varphi_\chi - (aH/k)v_\chi$ , and using the solutions in Eqs. (47,49) we can derive

$$\begin{aligned} \varphi_v(\mathbf{k}, t) &= C(\mathbf{k}) \left\{ 1 + \frac{K}{a^2} \frac{1}{4\pi G(\mu+p)} \left[ 1 - \frac{H}{a} \int_0^t a \left( 1 - \frac{K}{\dot{a}^2} \right) dt \right] \right\} + \frac{K}{a^2} \frac{H/a}{4\pi G(\mu+p)} d(\mathbf{k}) \\ &= C(\mathbf{k}) \left[ 1 + \frac{K}{a^2} \frac{H/a}{\mu+p} \int_0^t \frac{a(\mu+p)}{H^2} dt \right] + \frac{K}{a^2} \frac{H/a}{4\pi G(\mu+p)} d(\mathbf{k}). \end{aligned} \quad (53)$$

Thus, for  $K = 0$  (but for general  $\Lambda$ ) we have

$$\varphi_v(\mathbf{k}, t) = C(\mathbf{k}), \quad (54)$$

with the vanishing decaying solution. The disappearance of the decaying solution in Eq. (54) implies that the dominating decaying solutions in Eqs. (47,49) cancel out for  $K = 0$ . In fact, for  $K = 0$ , from Eqs. (12,13,14,17,18) we can derive

$$\frac{c_s^2 H^2}{a^3(\mu+p)} \left[ \frac{a^3(\mu+p)}{c_s^2 H^2} \dot{\varphi}_v \right] + c_s^2 \frac{k^2}{a^2} \varphi_v = \text{stresses}. \quad (55)$$

For a pressureless case ( $c_s^2 = 0$ ), instead of Eq. (55) we have  $\dot{\varphi}_v = 0$ ; see after Eq. (23). In the large-scale limit, and ignoring the stresses, we have an integral form solution

$$\varphi_v = C(\mathbf{x}) - \tilde{D}(\mathbf{x}) \int_0^t \frac{c_s^2 H^2}{a^3(\mu+p)} dt. \quad (56)$$

Compared with the solution in Eq. (54), the  $\tilde{D}$  term in Eq. (56) is  $c_s^2(k/aH)^2$  order higher than  $d$  terms in the other gauge. Therefore, for  $K = 0$ ,  $\varphi_v$  is conserved for the generally varying background equation of state, i.e., for general  $p = p(\mu)$ . Since the solutions in Eqs. (47,48,49) are valid in super-sound-horizon scale, the conservation property in Eq. (54) is valid in all scales in the matter dominated era.

For  $\varphi_\delta$ , by evaluating it in the comoving gauge we have

$$\varphi_\delta \equiv \varphi_v + \frac{\delta_v}{3(1+w)}. \quad (57)$$

Using Eqs. (53,48) we have

$$\varphi_\delta(\mathbf{k}, t) = C(\mathbf{k}) \left\{ 1 + \frac{k^2}{a^2} \frac{1}{12\pi G(\mu+p)} \left[ 1 - \frac{H}{a} \int_0^t a \left( 1 - \frac{K}{\dot{a}^2} \right) dt \right] \right\} + \frac{k^2}{a^2} \frac{H/a}{12\pi G(\mu+p)} d(\mathbf{k}). \quad (58)$$

Since the higher order term ignored in Eq. (53) is  $c_s^2(k/aH)^2$  order higher, in the medium with  $c_s \sim 1$  ( $\equiv c$ ), the terms involving  $(k/a)^2/(\mu+p)$  are not necessarily valid. Thus, to the leading order in the superhorizon scale we have

$$\varphi_\delta(\mathbf{k}, t) = C(\mathbf{k}). \quad (59)$$

Thus, in the superhorizon scale  $\varphi_\delta$  is conserved apparently considering  $K$  and  $\Lambda$ . [In the case of nonvanishing  $K$  we may need to be careful in defining the superhorizon scale. Here, as the superhorizon condition we simply took  $k^2$  term

to be negligible. In a hyperbolic (negatively curved) space with  $K = -1$ , since  $R^{(3)} = 6K/a^2$ , the distance  $a/\sqrt{|K|}$  introduces a curvature scale: on smaller scales the space is near flat and on larger scales the curvature term dominates. In the unit of the curvature scale  $k^2$  less than one ( $\sim |K|$ ) corresponds to the super-curvature scale, and  $k^2$  larger than one corresponds to the sub-curvature scale. Thus, when we ignored the  $k^2$  term compared with  $|K|$  in a negatively curved space, we are considering the super-curvature scale. In other words, for the hyperbolic backgrounds  $k^2$  takes continuous value with  $k^2 \geq 0$  where  $k^2 \geq 1$  corresponds to subcurvature mode whereas  $0 \leq k^2 < 1$  corresponds to supercurvature mode, and our large-scale limit took  $k^2 \rightarrow 0$ . A useful study in the hyperbolic situation can be found in [19].] However, in a medium with the negligible sound speed (like the matter dominated era), Eq. (58) is valid considering the  $k^2$  terms. Thus, as the scale enters the horizon in the matter dominated era  $\varphi_\delta$  wildly changes its behavior and is *no longer conserved inside the horizon*; inside the horizon  $\varphi_\delta$  is dominated by the  $\delta$  part which shows Newtonian behavior of  $\delta$  in most of the gauge conditions, see Table 1.

For  $\varphi_\kappa$ , by evaluating it in the uniform-density gauge and using Eq. (13), which gives  $\kappa_\delta = (k^2 - 3K)/(a^2 H)\varphi_\delta$ , we have

$$\varphi_\kappa = \varphi_\delta \left[ 1 + \frac{k^2 - 3K}{12\pi G(\mu + p)a^2} \right]^{-1}. \quad (60)$$

Thus, in the superhorizon scale and for  $K = 0$ ,  $\varphi_\kappa$  is conserved. As in  $\varphi_\delta$ , in a medium with the negligible sound speed, the solution in Eq. (60) with Eq. (58) is valid considering the  $k^2$  terms. Thus, as the scale enters the horizon in the matter dominated era  $\varphi_\kappa$  also changes its behavior and is *no longer conserved inside the horizon*.

Now we summarize: In the superhorizon scale  $\varphi_v$  and  $\varphi_\kappa$  are conserved for  $K = 0$ , while  $\varphi_\delta$  is conserved considering the general  $K$ . In the matter dominated era, for  $K = 0$ ,  $\varphi_v$  is still conserved in the super-sound-horizon scale which virtually covers all scales, whereas  $\varphi_\delta$  and  $\varphi_\kappa$  change their behaviors near the horizon crossing epoch and are no longer conserved inside the horizon. In this regard, we may say,  $\varphi_v$  is the *best conserved quantity*. Curiously, although  $\varphi_\chi$  is the variable most closely resembling the Newtonian gravitational potential, it is not conserved for changing equation of state, see Eq. (47); however, it is conserved independently of the horizon crossing in the matter dominated era for  $K = 0 = \Lambda$ , see Eq. (50).

Similar conservation properties of the curvature variable in certain gauge conditions remain true for models based on a minimally coupled scalar field and even on classes of generalized gravity theories. In the generalized gravity the uniform-field gauge is more suitable for handling the conservation property, and the uniform-field gauge coincides with the comoving gauge in the minimally coupled scalar field. Thorough studies of the minimally coupled scalar field and the generalized gravity, assuming  $K = 0 = \Lambda$ , are made in [16,20] and summarized in [21].

### C. Multi-component situation

We consider the energy-momentum tensor composed of multiple components as

$$T_{ab} = \sum_{(l)} T_{(l)ab}, \quad T_{(i)a;b}^b \equiv Q_{(i)a}, \quad \sum_{(l)} Q_{(l)a} = 0, \quad (61)$$

where  $i, l (= 1, 2, \dots, n)$  indicates  $n$  fluid components, and  $Q_{(i)a}$  considers possible mutual interactions among fluids. Since we are considering the scalar-type perturbation we decompose the  $Q_{(i)a}$  in the following way:

$$Q_{(i)0} \equiv -a(1 + \alpha)Q_{(i)}, \quad Q_{(i)} \equiv \bar{Q}_{(i)} + \delta Q_{(i)}, \quad Q_{(i)\alpha} \equiv J_{(i),\alpha}. \quad (62)$$

The scalar-type fluid quantities in Eq. (10) can be considered as the collective ones which are related to the individual component as:

$$\begin{aligned} \bar{\mu} &= \sum_{(l)} \bar{\mu}_{(l)}, \quad \bar{p} = \sum_{(l)} \bar{p}_{(l)}; \quad \delta\mu = \sum_{(l)} \delta\mu_{(l)}, \quad \delta p = \sum_{(l)} \delta p_{(l)}, \\ (\mu + p)v &= \sum_{(l)} (\mu_{(l)} + p_{(l)})v_{(l)}, \quad \sigma = \sum_{(l)} \sigma_{(l)}. \end{aligned} \quad (63)$$

For the background Eq. (11) remains valid for the collective fluid quantities. An additional equation follows from the individual energy-momentum conservation in Eq. (61) as

$$\dot{\mu}_{(i)} + 3H(\mu_{(i)} + p_{(i)}) = Q_{(i)}. \quad (64)$$

For the perturbations Eqs. (12-18) remain valid for the collective fluid quantities. The additional equations we need follow from the individual energy-momentum conservation in Eq. (61). From  $T_{(i)0;b}^b = Q_{(i)0}$  and using Eq. (12) we have the energy conservation of the fluid components

$$\delta\dot{\mu}_{(i)} + 3H(\delta\mu_{(i)} + \delta p_{(i)}) = -\frac{k}{a}(\mu_{(i)} + p_{(i)})v_{(i)} + \dot{\mu}_{(i)}\alpha + (\mu_{(i)} + p_{(i)})\kappa + \delta Q_{(i)}. \quad (65)$$

From  $T_{(i)\alpha;b}^b = Q_{(i)\alpha}$  we have the momentum conservation of the fluid components

$$\dot{v}_{(i)} + \left[ H(1 - 3c_{(i)}^2) + \frac{1 + c_{(i)}^2}{\mu_{(i)} + p_{(i)}} Q_{(i)} \right] v_{(i)} = \frac{k}{a} \left[ \alpha + \frac{1}{\mu_{(i)} + p_{(i)}} \left( \delta p_{(i)} - \frac{2}{3} \frac{k^2 - 3K}{a^2} \sigma_{(i)} - J_{(i)} \right) \right]. \quad (66)$$

By adding Eqs. (65,66) over the components we have Eqs. (17,18). In the multi-component situation Eqs. (12-18,65,66) are the complete set expressed in a gauge-ready form. If we decompose the pressure as  $\delta p_{(i)} = c_{(i)}^2 \delta\mu_{(i)} + e_{(i)}$  with  $c_{(i)}^2 \equiv \dot{p}_{(i)}/\dot{\mu}_{(i)}$ , comparing with Eq. (19) we have

$$e = \delta p - c_s^2 \delta\mu = \sum_{(l)} \left[ (c_{(l)}^2 - c_s^2) \delta\mu_{(l)} + e_{(l)} \right]. \quad (67)$$

Under the gauge transformation, from Eq. (20) in [8], we have:

$$\begin{aligned} \delta\tilde{\mu}_{(i)} &= \delta\mu_{(i)} - \dot{\mu}_{(i)}\xi^t, & \delta\tilde{p}_{(i)} &= \delta p_{(i)} - \dot{p}_{(i)}\xi^t, & \tilde{v}_{(i)} &= v_{(i)} - \frac{k}{a}\xi^t, \\ \tilde{J}_{(i)} &= J_{(i)} + Q_{(i)}\xi^t, & \delta\tilde{Q}_{(i)} &= \delta Q_{(i)} - \dot{Q}_{(i)}\xi^t. \end{aligned} \quad (68)$$

Thus, we have the following additional gauge conditions:

$$\delta\mu_{(i)} \equiv 0, \quad \delta p_{(i)} \equiv 0, \quad v_{(i)}/k \equiv 0, \quad \text{etc.} \quad (69)$$

Any one of these gauge conditions also fixes the temporal gauge condition completely. Studies of the multi-component situations can be found in [6,8,22].

#### D. Rotation and Gravitational Wave

For the vector-type perturbation, using Eqs. (9,10), we have:

$$8\pi G T_{\alpha}^{(v)0} = -\frac{\Delta + 2K}{2a^2} (B_{\alpha} + a\dot{C}_{\alpha}), \quad (70)$$

$$8\pi G \delta T_{\beta}^{(v)\alpha} = \frac{1}{2a^3} \left\{ a^2 \left[ B_{|\beta}^{\alpha} + B_{\beta}^{|\alpha} + a \left( C_{|\beta}^{\alpha} + C_{\beta}^{|\alpha} \right) \right] \right\}, \quad (71)$$

$$\frac{1}{a^3} \left( a^4 T_{\alpha}^{(v)0} \right)' = -\delta T_{\alpha|\beta}^{(v)\beta}, \quad (72)$$

where Eq. (72) follows from  $T_{\alpha;b}^b = 0$ , and apparently Eqs. (70,71) follow from the Einstein equations. We can show that Eq. (72) follows from Eqs. (70,71). We introduce a gauge-invariant velocity related variable  $V_{\alpha}^{(\omega)}(\mathbf{x}, t)$  as  $T_{\alpha}^{(v)0} \equiv (\mu + p)V_{\alpha}^{(\omega)}$ . We can show  $V_{\alpha}^{(\omega)} \propto a\omega$  where  $\omega$  is the amplitude of vorticity vector, see Sec. 5 of [8]. Thus, ignoring the anisotropic stress in the RHS of Eq. (72) we have the conservation of angular momentum as

$$\text{Angular Momentum} \sim a^3(\mu + p) \times a \times V_{\alpha}^{(\omega)}(\mathbf{x}, t) = \text{constant in time}, \quad (73)$$

which is valid for general  $K$ ,  $\Lambda$ , and  $p(\mu)$  in general scale.

For the tensor-type perturbation, using Eqs. (9,10) in the Einstein equations, we have

$$8\pi G \delta T_{\beta}^{(t)\alpha} = \ddot{C}_{\beta}^{\alpha} + 3H\dot{C}_{\beta}^{\alpha} - \frac{\Delta - 2K}{a^2} C_{\beta}^{\alpha}. \quad (74)$$

In the large-scale limit, assuming  $K = 0$  and ignoring the anisotropic pressure Eq. (74) has a general integral form solution

$$C_{\beta}^{\alpha}(\mathbf{x}, t) = C_{1\beta}^{\alpha}(\mathbf{x}) - D_{\beta}^{\alpha}(\mathbf{x}) \int^t \frac{1}{a^3} dt, \quad (75)$$

where  $C_{1\beta}^{\alpha}$  and  $D_{\beta}^{\alpha}$  are integration constants for relatively growing and decaying solutions, respectively. Thus, ignoring the transient term the amplitude of the gravitational wave is conserved in the super-horizon scale considering general evolution of the background equation of state.

The conservation of angular momentum and the equation for the gravitational wave were first derived in [1]. Evolutions in the multi-component situation were considered in Sec. 5 of [8].

## V. DISCUSSIONS

We would like to make comments on related works in several *textbooks*: Eq. (15.10.57) in [12], Eq. (10.118) in [23], Eq. (11.5.2) in [24], Eq. (8.52) in [25], and problem 6.10 in [26] are in error. All these errors are essentially about the same point involved with a fallible algebraic mistake in the synchronous gauge. The correction in the case of [12] was made in [27]: in a medium with a nonvanishing pressure the equation for the density fluctuation in the synchronous gauge becomes third order because of the presence of a gauge mode in addition to the physical growing and decaying solutions which is true even in the large-scale limit. The truncated second order equation in [12,25] picks up a gauge mode instead of the physical decaying solution in the synchronous gauge. The errors in [23,26] are based on imposing the synchronous gauge and the comoving gauge simultaneously, and thus happen to end up with the same truncated second order equation as in [12]. In a medium with nonvanishing pressure one cannot impose the two gauge conditions simultaneously (even in the large-scale limit). In Sec. 11.5 of [24] the authors proposed a simple modification of the Newtonian theory which again happens to end up with the same incorrect equation as in the other books. In the Appendix we elaborate our point. For comments on other related errors in the literature, see Sec. 3.10 in [15] and [27].

We would like to remark that these errors found in the synchronous gauge are not due to any esoteric aspect of the gauge choice in the relativistic perturbation analyses. Although not fundamental, the number of errors found in the literature concerning the synchronous gauge seem to indicate the importance of using the proper gauge condition in handling the problems. For our own choice of the preferred gauge-invariant variables suitable for handling the hydrodynamic perturbation and consequent analyses, see Sec. IV; the reasons for such choices are made in Sec. III. We wish to recall, however, that although one may need to do more algebra in tracing the remnant gauge mode, even the synchronous gauge condition is adequate for handling many problems as was carefully done in the original study by Lifshitz in 1946 [1].

In this paper we have tried to identify the variables in the relativistic perturbation analysis which reproduce the correct Newtonian behavior in the pressureless limit. In the first part we have shown that  $\delta$  in the comoving gauge ( $\delta_v$ ) and  $v$  and  $\varphi$  in the zero-shear gauge ( $v_{\chi}$  and  $\varphi_{\chi}$ ) show the same behavior as the corresponding Newtonian variables *in general scales*. In fact, these results have already been presented in [10]. Also, various general and asymptotic solutions for every variable in the pressureless medium are presented in the Tables of [10]. Compared with [10], in Sec. III we tried to reinforce the correspondence by explicitly showing the second order differential equations in several gauge conditions. We have also added some additional insights gained after publishing [10]. The second part contains some original results. Using the gauge-invariant variables  $\delta_v$ ,  $v_{\chi}$  and  $\varphi_{\chi}$  we write the relativistic hydrodynamic cosmological perturbation equations in Eqs. (41-44); actually, these equations are also known in [5]. The new results in this paper are the compact way of deriving the general large-scale solutions in Eqs. (47-49), and the clarification of the general large-scale conservation property of  $\varphi_v$  in Eq. (53).

The underlying mathematical or physical reasons for the variables  $\delta_v$ ,  $\varphi_{\chi}$ ,  $v_{\chi}$ , and  $\varphi_v$  having the distinguished behaviors, compared with many other available gauge-invariant combinations, may still deserve further investigation.

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## APPENDIX: COMMON ERRORS IN THE SYNCHRONOUS GAUGE

In the following we correct a minor confusion in the literature concerning perturbation analyses in the synchronous gauge. The argument is based on [27]. For simplicity, we consider a situation with  $K = 0 = \Lambda$ ,  $e = 0 = \sigma$  and  $w = \text{constant}$  (thus  $c_s^2 = w$ ).

The equation for the density perturbation in the *comoving gauge* is given in Eq. (45). In our case we have

$$\ddot{\delta}_v + (2 - 3w)H\dot{\delta}_v + \left[ c_s^2 \frac{k^2}{a^2} - 4\pi G\mu(1 - w)(1 + 3w) \right] \delta_v = 0. \quad (A1)$$

The solution in the super-sound-horizon scale is presented in Eq. (50).

In the *synchronous gauge*, from Eqs. (16-18) we have (the subindex  $\alpha$  indicates the synchronous gauge variable)

$$\ddot{\delta}_\alpha + 2H\dot{\delta}_\alpha + \left[ c_s^2 \frac{k^2}{a^2} - 4\pi G\mu(1 + w)(1 + 3w) \right] \delta_\alpha + 3w(1 + w) \frac{k}{a} H v_\alpha = 0, \quad (A2)$$

$$\dot{v}_\alpha + (1 - 3w)Hv_\alpha - \frac{k}{a} \frac{w}{1 + w} \delta_\alpha = 0. \quad (A3)$$

Notice that for  $w \neq 0$  these two equations are generally coupled even in the large-scale limit. From these two equations we can derive a third order differential equation for  $\delta_\alpha$  as

$$\begin{aligned} \delta_\alpha''' + \frac{11 - 3w}{2} H \ddot{\delta}_\alpha + \left( \frac{5 - 24w - 9w^2}{2} H^2 + c_s^2 \frac{k^2}{a^2} \right) \dot{\delta}_\alpha \\ + \left[ \frac{3}{4} (1 + w)(1 + 3w)(-1 + 9w) H^3 + \frac{3}{2} (1 + w) H c_s^2 \frac{k^2}{a^2} \right] \delta_\alpha = 0. \end{aligned} \quad (A4)$$

In the large-scale limit the solutions are

$$\delta_\alpha \propto t^{\frac{2(1+3w)}{3(1+w)}}, \quad t^{\frac{9w-1}{3(1+w)}}, \quad t^{-1}. \quad (A5)$$

In Appendix B of [27] we showed that the third solution with  $\delta_\alpha \propto t^{-1}$  is nothing but a gauge mode for a medium with  $w \neq 0$ . [As mentioned before, the combination  $\delta_\alpha$  is not gauge-invariant. From Eq. (20) we have  $\delta_\alpha \equiv \delta + 3H(1 + w) \int^t \alpha dt$  and the lower bound of the integration gives rise to the gauge mode which is proportional to  $t^{-1}$ .] For a pressureless case the physical decaying solution also behaves as  $t^{-1}$  and the second solution in Eq. (A5) is invalid, see Sec. 4 in [27]. In Eq. (B5) of [27] we derived the relation between solutions in the two gauges explicitly; the growing solutions are the same in both gauges whereas the decaying solutions differ by a factor  $(k/aH)^2 \propto t^{\frac{2(1+3w)}{3(1+w)}}$ .

Now, we would like to point out that, in a medium with  $w \neq 0$ , one cannot ignore the last term in Eq. (A2) even in the large-scale limit. If we *incorrectly* ignore the last term in Eq. (A2) we recover the *wrong equation* in the textbooks mentioned in Sec. V which is

$$\ddot{\delta}_\alpha + 2H\dot{\delta}_\alpha + \left[ c_s^2 \frac{k^2}{a^2} - 4\pi G\mu(1 + w)(1 + 3w) \right] \delta_\alpha = 0. \quad (A6)$$

In the large-scale we have solutions

$$\delta_\alpha \propto t^{\frac{2(1+3w)}{3(1+w)}}, \quad t^{-1}. \quad (A7)$$

By ignoring the last term in Eq. (A2), in the large-scale limit we happen to recover the fictitious gauge mode under the price of losing the physical decaying solution [the second solution in Eq. (A5)]. Thus, in the large-scale limit one cannot ignore the last term in Eq. (A2) in a medium with a general pressure; the reason is obvious if we see Eq. (A4). Also, one cannot impose both the synchronous gauge condition and the comoving gauge condition simultaneously. If we simultaneously impose such two conditions, thus setting  $v \equiv 0 \equiv \alpha$ , from Eq. (18) we have

$$0 = \frac{k}{a(1 + w)} \left( c_s^2 \delta + \frac{e}{\mu} - \frac{2}{3} \frac{k^2 - 3K}{a^2} \frac{\sigma}{\mu} \right). \quad (A8)$$

Thus, for  $e = 0 = \sigma$  and medium with the nonvanishing pressure we have  $\delta = 0$  which is a meaningless system; this argument remains valid even in the large-scale limit.

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**Table 1. Newtonian correspondences:** For the synchronous gauge we ignore the gauge mode. Thus the synchronous gauge is equivalent to the comoving gauge. X indicates that the behavior differs from the Newtonian one. The small-scale implies the scale smaller than the (visual) horizon. Explicit forms of exact and asymptotic solutions for every variable are presented in [10,15].

Gauge	Variable	General Scale	Small Scale
Comoving gauge	$\delta_v/k$	Newtonian	Newtonian
Zero-shear gauge	$\delta_\chi$	X	Newtonian
Uniform-expansion gauge	$\delta_\kappa$	X	Newtonian
Synchronous gauge	$\delta_\alpha$	Newtonian	Newtonian
Uniform-curvature gauge	$\delta_\varphi$	X	Newtonian
Uniform-density gauge	$\delta \equiv 0$	0	0
Comoving gauge	$v \equiv 0$	0	0
Zero-shear gauge	$v_\chi$	Newtonian	Newtonian
Uniform-expansion gauge	$v_\kappa$	X	Newtonian
Synchronous gauge	$v_\alpha$	0	0
Uniform-curvature gauge	$v_\varphi$	X	X
Uniform-density gauge	$v_\delta$	X	X
Comoving gauge	$\varphi_v$	X	X
Zero-shear gauge	$\varphi_\chi$	Newtonian	Newtonian
Uniform-expansion gauge	$\varphi_\kappa$	X	Newtonian
Synchronous gauge	$\varphi_\alpha$	X	X
Uniform-curvature gauge	$\varphi \equiv 0$	0	0
Uniform-density gauge	$\varphi_\delta$	X	X

**Note added:**

We introduce a variable

$$\Phi \equiv \varphi_v - \frac{K/a^2}{4\pi G(\mu + p)} \varphi_\chi. \quad (B.1)$$

From the general large-scale solutions in eqs. (47),(53) we can show

$$\Phi(\mathbf{x}, t) = C(\mathbf{x}), \quad (B.2)$$

where the dominating decaying solution vanished. Thus, remarkably,  $\Phi$  is conserved, in the limit of vanishing  $c_s^2 k^2/a^2$  term, considering general  $K$ ,  $\Lambda$ , and time-varying  $p(\mu)$ . In a pressureless medium eq. (B.2) is an exact solution valid in general scale.